

A SEPTIC WITH 99 REAL NODES

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ABSTRACT. We find a surface of degree 7 in $\mathbb{P}^3(\mathbb{R})$ with 99 real nodes within a family of surfaces with dihedral symmetry: First, we consider this family over some small prime fields, which allows us to test all possible parameter sets using computer algebra. In this way we find some examples of 99-nodal surfaces over some of these finite fields. Then, the examination of the geometry of these surfaces allows us to determine the parameters of a 99-nodal septic in characteristic zero. This narrows the possibilities for $\mu(7)$, the maximum number of nodes on a septic, to: $99 \leq \mu(7) \leq 104$. When reducing our surface modulo 5, we even obtain a 100-nodal septic in $\mathbb{P}^3(\mathbb{F}_5)$.

INTRODUCTION

We denote by $\mu(d)$ the maximum number of nodes (i.e. singularities of type A_1 , also called ordinary double points) a surface of degree d in $\mathbb{P}^3(\mathbb{C})$ can have. The restrictions on $\mu(d)$ known so far are summarized in the following table:

degree	2	3	4	5	6	7	8	9	10	11	12	d
$\mu(d) \geq$	1	4	16	31	65	93	168	216	345	425	600	$\frac{5}{12}d^3$
$\mu(d) \leq$	1	4	16	31	65	104	174	246	360	480	645	$\frac{4}{9}d^3$

In this article we show:

$$(1) \quad \mu(7) \geq 99.$$

From the table above, we see in particular that $\mu(d)$ is known up to $d = 6$. The upper bound $\mu(7) \leq 104$ is given by Varchenko's spectrum bound [14]. Note that for $d = 7$ Miyaoka's bound [11] is 112, but Givental's bound [6] also gives 104.

The previously known septic with the greatest number of nodes was the example of Chmutov [2] with 93 nodes: It comes from a construction that works for any degree d . For $d \leq 5$ and the even degrees $d = 6, 8, 10, 12$ there are examples exceeding Chmutov's lower bound: [1], [4], [12]. These had been obtained by using some beautiful geometric arguments based on Segre's idea [13].

In this note, we explain how to use computer algebra experiments over prime fields to treat the case $d = 7$ and to find the first surface of odd degree greater than 5 that exceeds Chmutov's general lower bound. Given an explicit equation of a family of hypersurfaces, there is in fact an algorithm to find those examples with the greatest number of nodes: We already applied this successfully in [10], but because of computer performance restrictions we cannot use this technique in the present case. Instead, we choose a more geometric approach to study the family.

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1. THE FAMILY

Inspired by many authors (see in particular: [1], [3], [4]), we look for septic surfaces with many nodes in $\mathbb{P}^3(\mathbb{C})$ within a 7-parameter family of surfaces $S_{a_1, a_2, \dots, a_7} := P - U_{a_1, a_2, \dots, a_7}$ of degree 7 admitting the dihedral symmetry D_7 of a 7-gon:

$$\begin{aligned} P &:= 2^6 \cdot \prod_{j=0}^6 \left[\cos\left(\frac{2\pi j}{7}\right) x + \sin\left(\frac{2\pi j}{7}\right) y - z \right] \\ &= x \cdot [x^6 - 3 \cdot 7 \cdot x^4 y^2 + 5 \cdot 7 \cdot x^2 y^4 - 7 \cdot y^6] \\ &\quad + 7 \cdot z \cdot [(x^2 + y^2)^3 - 2^3 \cdot z^2 \cdot (x^2 + y^2)^2 + 2^4 \cdot z^4 \cdot (x^2 + y^2)] - 2^6 \cdot z^7, \\ U_{a_1, a_2, \dots, a_7} &:= (z + a_5 w) (a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3 + (a_6 z + a_7 w)(x^2 + y^2))^2. \end{aligned}$$

P is the product of 7 planes in $\mathbb{P}^3(\mathbb{C})$ meeting in the point $(0 : 0 : 0 : 1)$ and admitting D_7 -symmetry with rotation axes $\{x = y = 0\}$: In fact, P is invariant under the map $y \mapsto -y$ and $P \cap \{z = z_0\}$ is a regular 7-gon for $z_0 \neq 0$. U is also D_7 -symmetric, because x and y only appear as $x^2 + y^2$.

The generic surface S has nodes at the $3 \cdot 21 = 63$ intersections of the $\binom{7}{2} = 21$ doubled lines of P with the doubled cubic of U . We look for parameters a_1, a_2, \dots, a_7 , s.t. the corresponding surface has 99 nodes.

As $S_{a_1, a_2, \dots, a_7}(x, y, z, \lambda w) = S_{a_1, \lambda a_2, \lambda^2 a_3, \lambda^3 a_4, \lambda a_5, a_6, \lambda a_7}(x, y, z, w) \ \forall \lambda \in \mathbb{C}^*$, we choose $a_7 := 1$. Moreover, experiments over prime fields suggest that the maximum number of nodes on such surfaces is 99 and that such examples exist for $a_6 = 1$. As we are mainly interested in finding an example with 99 nodes, we restrict ourselves to the sub-family:

$$S := S_{a_1, a_2, a_3, a_4, a_5, 1, 1} = P - U_{a_1, a_2, a_3, a_4, a_5, 1, 1}.$$

Some other cases, e.g. $a_6 = 0$, also lead to 99-nodal septic surfaces; this will be discussed elsewhere.

2. REDUCTION TO THE CASE OF PLANE CURVES

To simplify the problem of locating examples with 99 nodes within our family S , we restrict our attention to the $\{y = 0\}$ -plane and search for plane curves $S|_{y=0}$ (we write S_y for short) with many nodes. This is possible because of the symmetry of the construction (see [3, p. 18, cor. 2.3.10] for details):

Lemma 1. *A member $S = S_{a_1, a_2, a_3, a_4, a_5, 1, 1}$ of our family of surfaces has only ordinary double points as singularities, if $(1 : i : 0 : 0) \notin S$ and the surface does only contain ordinary double points as singularities in the plane $\{y = 0\}$. If the plane septic S_y has exactly n nodes and if exactly n_{xy} of these nodes are on the axes $\{x = y = 0\}$ then the surface S has exactly $n_{xy} + 7 \cdot (n - n_{xy})$ nodes and no other singularities. Each singularity of S_y which is not on $\{x = y = 0\}$ gives an orbit of 7 singularities of S under the action of the dihedral group D_7 .*

To understand the geometry of the plane septic S_y better, we look at the singularities that occur for generic values of the parameters. First, we compute:

$$\begin{aligned}
 P|_{y=0} &= x^7 + 7 \cdot x^6 z - 7 \cdot 2^3 \cdot x^4 z^3 + 7 \cdot 2^4 \cdot x^2 z^5 - 2^6 \cdot z^7 \\
 &= \frac{(x-z)^2}{2^4} \cdot \underbrace{(x + (-\rho)z)^2}_{=:L_1} \cdot \underbrace{(2x + (\rho^2 + 4\rho)z)^2}_{=:L_2} \cdot \underbrace{(2x + (-\rho^2 - 2\rho + 8)z)^2}_{=:L_3}, \\
 U|_{y=0} &= (z + a_5 w) \underbrace{((z+w)x^2 + a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3)^2}_{=:C},
 \end{aligned}$$

where ρ satisfies:

$$(2) \quad \rho^3 + 2^2 \rho^2 - 2^2 \rho - 2^3 = 0.$$

The three points G_{ij} of intersection of C with the line L_i are ordinary double points of the plane septic $S_y = P|_{y=0} - U|_{y=0}$ for generic values of the parameters, s.t. we have $3 \cdot 3 = 9$ *generic* singularities (see fig. 1).

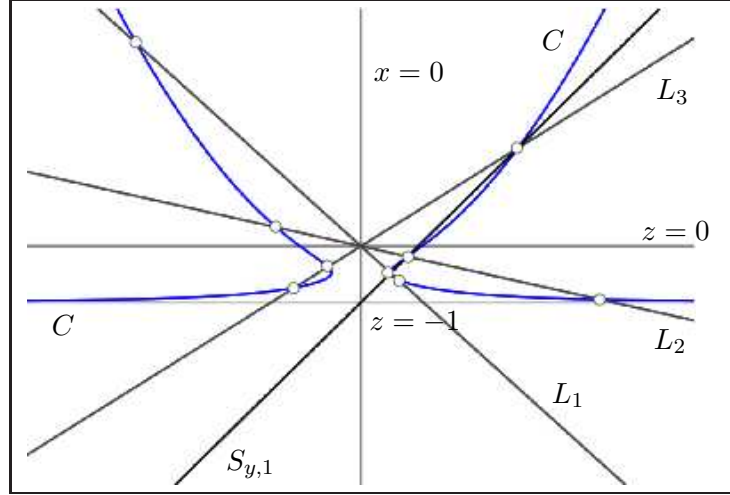


FIGURE 1. The three doubled lines L_i and the doubled cubic C intersect in $3 \cdot 3 = 9$ points G_{ij} . These are the *generic* singularities of the plane septic S_y .

3. FINDING SOLUTIONS OVER SOME PRIME FIELDS

By running over all possible parameter combinations over some small prime fields \mathbb{F}_p using the computer algebra system SINGULAR [8], we find some 99-nodal surfaces over these fields: For a given set of parameters a_1, a_2, \dots, a_5 , we can easily check the actual number of nodes on the corresponding surface using computer algebra (see [7, appendix A, p. 487]).

As indicated in the previous section, we work in the plane $\{y = 0\}$ for faster computations. It turns out that the greatest number of nodes on S_y is 15 over the small prime fields \mathbb{F}_p , $11 \leq p \leq 53$: See table 1 on page 5. The prime fields \mathbb{F}_p , $2 \leq p \leq 7$, are not listed because they are special cases: These primes appear as coefficients or exponents in the equation of our family. In each of the cases

we checked, one of the 15 singular points lies on the axes $\{x = 0\}$, such that the corresponding surface has exactly $14 \cdot 7 + 1 = 99$ nodes and no other singularities.

4. THE GEOMETRY OF THE 15-NODAL SEPTIC PLANE CURVE

To find parameters a_1, a_2, \dots, a_5 in characteristic 0 we want to use geometric properties of the 15-nodal septic plane curve S_y . But as we do not know any such property yet, we use our prime field examples to get some good ideas:

Observation 1. *In all our prime field examples of 15-nodal plane septics S_y , we have:*

- (1) S_y splits into a line $S_{y,1}$ and a sextic $S_{y,6}$: $S_y = S_{y,1} \cdot S_{y,6}$. The plane curve $S_{y,6}$ of degree 6 has $15 - 6 = 9$ singularities. Note that this property is similar to the one of the 31-nodal D_5 -symmetric quintic in $\mathbb{P}^3(\mathbb{C})$ constructed by W. Barth: See [3, p. 27-32] for a description.

The line and the sextic have some interesting geometric properties (see fig. 1 on the preceding page and fig. 3 on page 8):

- (2) $S_{y,1} \cap S_{y,6} = \{R, G_{1j_1}, G_{2j_2}, G_{3j_3}, O_1, O_2\}$, where R is a point on the axes $\{x = 0\}$ and the G_{ij_k} are three of the 9 generic singularities G_{ij} of S_y , one on each line L_i , and O_1, O_2 are some other points that neither lie on $\{x = 0\}$, nor on one of the L_i .
- (3) The sextic $S_{y,6}$ has the six generic singularities G_{ij} , $(i, j) \in \{1, 2, 3\}^2 \setminus \{(1, j_1), (2, j_2), (3, j_3)\}$, and three exceptional singularities: E_1, E_2, E_3 .

In many prime field experiments, we have furthermore:

- (4) In the projective x, z, w -plane, the point R has the coordinates $(0 : -1 : 1)$, s.t. the line $S_{y,1}$ has the form $S_{y,1} : z + t \cdot x + w = 0$ for some parameter t (see also table 1 on the facing page).

The other cases ($R = (0 : c : 1), c \neq -1$) lead to more complicated equations and will not be discussed here.

Using this observation as a guess for our septic in characteristic 0, we obtain several polynomial conditions on the parameters. Using SINGULAR to eliminate variables, we find the following relation between the parameters a_4 and t :

$$(3) \quad t \cdot \underbrace{(a_4 t^3 + t)}_{=: \alpha}^2 + t - 1 = 0,$$

which can be parametrized by α : $t = -\frac{1}{1+\alpha^2}$, $a_4 = (\alpha(1+\alpha^2) - 1)(1+\alpha^2)^2$. Further eliminations allow us to express all the other parameters in terms of α :

- $a_1 = \alpha^7 + 7\alpha^5 - \alpha^4 + 7\alpha^3 - 2\alpha^2 - 7\alpha - 1$,
- $a_2 = (\alpha^2 + 1)(3\alpha^5 + 14\alpha^3 - 3\alpha^2 + 7\alpha - 3)$,
- $a_3 = (\alpha^2 + 1)^2(3\alpha^3 + 7\alpha - 3)$,
- $a_5 = -\frac{\alpha^2}{1+\alpha^2}$.

5. THE 1-PARAMETER FAMILY OF PLANE SEXTICS

We use once more our explicit examples of 15-nodal septic plane curves over prime fields to finally be able to write down a condition for α in characteristic 0.

First, note that we can now easily obtain the equation of $S_{y,6}$ by dividing the equation of our septic curve S_y by the equation of the line $S_{y,1} = z + tx + w = z - \frac{1}{1+\alpha^2}x + w$. $S_{y,6}$ is a sextic which has 6 nodes for generic α , but should have 9

Field	a_1	a_2	a_3	a_4	a_5	$S_{y,1}$	α
\mathbb{F}_{11}	2	3	5	2	-5	$z = x - w$	$\alpha = -3$
\mathbb{F}_{19}	-7	-2	7	1	8	$z = 8x - w$	$\alpha = 7$
\mathbb{F}_{19}	2	0	1	9	7	$z = 9x - w$	$\alpha = -4$
\mathbb{F}_{19}	5	-9	7	-3	-1	$z = 2x - w$	$\alpha = -3$
\mathbb{F}_{23}	-5	11	10	1	7	$z = -9x - w$	$\alpha = -2$
\mathbb{F}_{31}	-15	-13	-5	13	-10	$z = -2x - w$	$\alpha = -13$
\mathbb{F}_{31}	1	-2	14	-9	11	$z = 15x - w$	$\alpha = -11$
\mathbb{F}_{31}	14	-10	-13	-14	-11	$z = -13x - w$	$\alpha = -7$
\mathbb{F}_{43}	-11	15	0	-13	-6	$z = -6x - w$	$\alpha = 7$
\mathbb{F}_{43}	20	16	-1	-14	10	$z = -12x - w$	$\alpha = 14$
\mathbb{F}_{43}	-9	3	-3	-11	5	$z = 18x - w$	$\alpha = -21$
\mathbb{F}_{53}	-8	20	14	18	11	$z = 25x - w$	$\alpha = 4$
\mathbb{F}_{53}	-2	-10	-14	-26	16	$z = -9x - w$	$\alpha = 24$
\mathbb{F}_{53}	10	25	-4	22	25	$z = -16x - w$	$\alpha = 25$

TABLE 1. A few examples of parameters giving 15-nodal septic plane curves (and 99-nodal surfaces) over prime fields (see [9] for more exhaustive tables).

double points for some special values of α . One idea to determine these particular values is to find a geometric relation between the 6 generic singular points and the 3 exceptional ones.

5.1. Three Conics. Looking at the equations describing the singular points of our examples of 9-nodal sextics $S_{y,6}$ over the prime fields, we see the following:

Observation 2. *For all our 9-nodal examples of plane sextics over prime fields, there are three conics through six of these points each (see fig. 2 on the next page):*

- (1) one conic C_0 through the 6 generic singularities,
- (2) one conic C_1 through the 3 exceptional singularities and 3 of the generic ones,
- (3) one conic C_2 through the 3 exceptional singularities and the other 3 generic ones.

Moreover, the three conics have the following properties over the prime fields:

- (4) C_1 has the form:

$$(4) \quad C_1 : x^2 + kz^2 + (k+4)zw = 0,$$

where k is a still unknown parameter. In particular, C_1 is symmetric with respect to $x \mapsto -x$ and contains the point $(0 : 0 : 1)$.

- (5) C_0 intersects the other two conics on the $\{x = 0\}$ -axes (see fig. 2 on the next page):

$$(5) \quad X_1 := C_0 \cap C_1 \cap \{x = 0\}, \quad X_2 := C_0 \cap C_2 \cap \{x = 0\}.$$

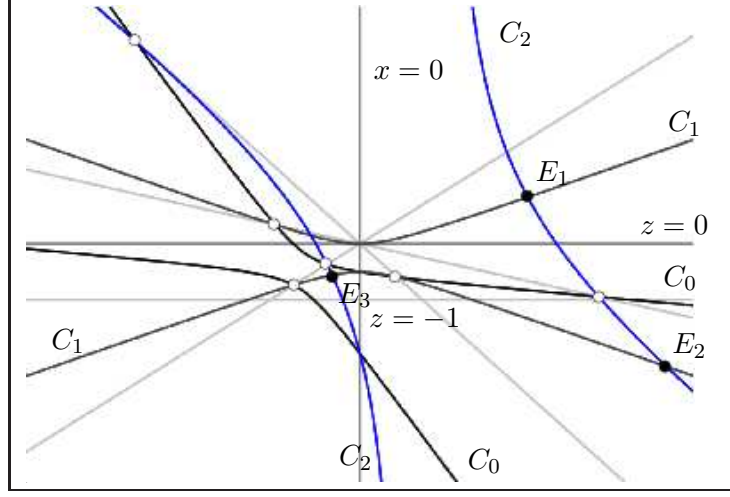


FIGURE 2. Three conics relating the 9 double points of the sextic $S_{y,6}$. E_1, E_2 , and E_3 (black) are the exceptional singularities (i.e. they do not lie on one of the lines L_i , see fig. 1 on page 3). The white points are the generic singularities, coming from the intersection of the doubled cubic C with the three doubled lines L_i .

To determine the new parameter k in equation (4), we will use (5). We compute the two points of C_0 on the $\{x = 0\}$ -axes explicitly using SINGULAR: First, the ideal $I_{S_{y,6}}^{gen}$ describing the six generic singularities of $S_{y,6}$ can be computed from the ideal $I_{S_y}^{gen} := (C, L_1 L_2 L_3)$ describing the 9 generic singularities of S_y by calculating the following ideal quotient: $I_{S_{y,6}}^{gen} = I_{S_y}^{gen} : S_{y,1}$. Now, the equation of C_0 can be obtained by taking the degree-2-part of the ideal $I_{S_{y,6}}^{gen}$:

$$(6) \quad \begin{aligned} C_0 : \quad & \alpha x^2 + (\alpha^3 + 5\alpha - 1)xz + (\alpha^5 + (\alpha^3 + \alpha - 1)xw \\ & + 6\alpha^3 - \alpha^2 + \alpha - 1)z^2 + (2\alpha^5 + 8\alpha^3 - 2\alpha^2 + 6\alpha - 2)zw \\ & + (\alpha^5 + 2\alpha^3 - \alpha^2 + \alpha - 1)w^2 = 0. \end{aligned}$$

Thus, $\{P^+, P^-\} := C_0 \cap \{x = 0\} = \left\{ \left(0 : \frac{-2(\alpha^3 + 3\alpha - 1)(1 + \alpha^2) \pm \beta(\alpha)}{2(\alpha^5 + 6\alpha^3 - \alpha^2 + \alpha - 1)} : 1 \right) \right\}$, where

$$(7) \quad \begin{aligned} \beta(\alpha)^2 &:= (\alpha^3 + 3\alpha - 1)^2(1 + \alpha^2)^2 \\ &\quad - 4(\alpha^5 + 6\alpha^3 - \alpha^2 + \alpha - 1)(1 + \alpha^2)(\alpha^3 + \alpha - 1) \\ &= 16\alpha(2\alpha^5 + 4\alpha^3 - \alpha^2 + 2\alpha - 1). \end{aligned}$$

C_1 intersects the $\{x = 0\}$ -axes in exactly two points: $(0 : 0 : 1)$ and X_1 . Hence, we can determine the two possibilities for the parameter $k \in \mathbb{Q}(\alpha, \beta(\alpha))$ in equation (4) for C_1 : Together with the z and w -coordinates of the points P^\pm , $C_1 \cap \{x = 0\} = \{kz^2 + kzw + 4zw = 0\}$ leads to the following two possibilities:

$$(8) \quad C_1 : \quad x^2 + \frac{-4P_z^\pm}{P_z^\pm(P_z^\pm + 1)}z(z + w) + 4zw = 0.$$

5.2. The Condition on α . The equations of the conics C_0 and C_1 will allow us to compute the condition on α , s.t. the sextic $S_{y,6}$ has 9 singularities, using the following (see observation 2 and fig. 2):

- C_0 intersects the three doubled lines L_i exactly in the six generic singularities.
- C_1 intersects the three doubled lines L_i exactly in three of these six generic singularities and the origin (which counts three times).

Thus, the set of z -coordinates of the three points $(C_1 \cap L_1 L_2 L_3) \setminus \{(0 : 0 : 1)\}$ has to be contained in the set of z -coordinates of the six points $C_0 \cap L_1 L_2 L_3$. This means that the remainder q of the following division (res_x denotes the resultant with respect to x)

$$(9) \quad res_x(C_0, L_1 L_2 L_3) = p(z) \cdot \left(\frac{1}{z^3} \cdot res_x(C_1, L_1 L_2 L_3) \right) + q(z)$$

should vanish: $q = 0$.

As the degree of the remainder is $\deg(q) = 2$, this gives 3 conditions on α and $\beta(\alpha)$, coming from the fact that all the 3 coefficients of $q(z)$ have to vanish. It turns out that it suffices to take one of these, the coefficient of z^2 , which can be written in the form $c(\alpha) + \beta(\alpha)d(\alpha)$, where $c(\alpha)$ and $d(\alpha)$ are polynomials in $\mathbb{Q}[\alpha]$. As a condition on α only we can take:

$$cond(\alpha) := (c(\alpha) + \beta(\alpha)d(\alpha)) \cdot (c(\alpha) - \beta(\alpha)d(\alpha)) \in \mathbb{Q}[\alpha],$$

which is of degree 150.

This condition $cond(\alpha)$ vanishes for those α for which the corresponding surface has 99 nodes and for several other α . To obtain a condition which exactly describes those α we are looking for, we factorize $cond(\alpha) = f_1 \cdot f_2 \cdots f_k$ (e.g., using SINGULAR again). Substituting in each of these factors our solutions over the prime fields, we see that the only factor that vanishes is: $7\alpha^3 + 7\alpha + 1 = 0$.

6. THE EQUATION OF THE 99-NODAL SEPTIC

Up to this point, it is still only a guess — verified over some prime fields — that the values α satisfying the condition above give 99-nodal septics in characteristic 0. But a straightforward computation with SINGULAR shows:

Theorem 1 (99-nodal Septic). *Let $\alpha \in \mathbb{C}$ satisfy:*

$$(10) \quad 7\alpha^3 + 7\alpha + 1 = 0.$$

Then the surface S_α in $\mathbb{P}^3(\mathbb{C})$ of degree 7 with equation $S_\alpha := P - U_\alpha$ has exactly 99 ordinary double points and no other singularities, where

$$\begin{aligned} P &:= x \cdot [x^6 - 3 \cdot 7 \cdot x^4 y^2 + 5 \cdot 7 \cdot x^2 y^4 - 7 \cdot y^6] \\ &\quad + 7 \cdot z \cdot [(x^2 + y^2)^3 - 2^3 \cdot z^2 \cdot (x^2 + y^2)^2 + 2^4 \cdot z^4 \cdot (x^2 + y^2)] - 2^6 \cdot z^7, \\ U_\alpha &:= (z + a_5 w) ((z + w)(x^2 + y^2) + a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3)^2, \\ a_1 &:= -\frac{12}{7}\alpha^2 - \frac{384}{49}\alpha - \frac{8}{7}, & a_2 &:= -\frac{32}{7}\alpha^2 + \frac{24}{49}\alpha - 4, \\ a_3 &:= -4\alpha^2 + \frac{24}{49}\alpha - 4, & a_4 &:= -\frac{8}{7}\alpha^2 + \frac{8}{49}\alpha - \frac{8}{7}, \\ a_5 &:= 49\alpha^2 - 7\alpha + 50. \end{aligned}$$

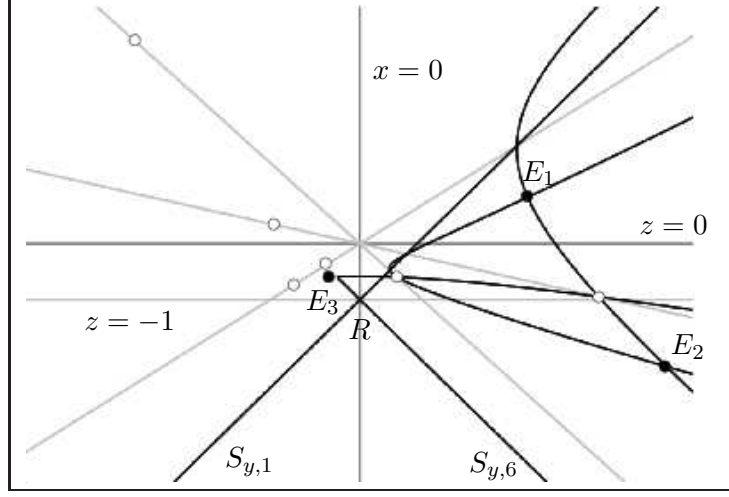


FIGURE 3. The 15-nodal plane septic $S_{y_{\alpha_R}} = S_{y,1_{\alpha_R}} \cdot S_{y,6_{\alpha_R}}$ (see (11) on page 8); the singularities of the sextic $S_{y,6_{\alpha_R}}$ are marked by large circles: The three exceptional singularities E_1, E_2, E_3 are marked in black, the generic singularities in white. The five left-most nodes are real isolated ones. Only five of the six intersections of the line $S_{y,1_{\alpha_R}}$ and the sextic $S_{y,6_{\alpha_R}}$ are visible because we just show a small part of the whole (x, z) -plane.

There is exactly one real solution $\alpha_R \in \mathbb{R}$ to the condition (10),

$$(11) \quad \alpha_R \approx -0.14010685,$$

and all the singularities of S_{α_R} are also real.

Proof. To show that the surface has 99 nodes and no other singularities we verify the hypothesis of lemma 1 on page 2, because this speeds up the computations. The algorithm is straightforward. Our SINGULAR code is listed in the appendix and can also be obtained from [9].

Using the fact that $\beta(\alpha)^2 = \left(\frac{12}{7}\right)^2$ together with the geometric description of the singularities of the plane septic given in the previous sections, it is also straightforward to verify the reality assertion. \square

7. CONCLUDING REMARKS

The existence of the real α_R allows us to use the program SURF [5] to compute an image of the 99-nodal septic S_{α_R} (fig. 4 on the next page). When denoting the maximum number of real singularities a septic in $\mathbb{P}^3(\mathbb{R})$ can have by $\mu^R(7)$, we get, with the remarks mentioned in the introduction:

Corollary 2.

$$99 \leq \mu^R(7) \leq \mu(7) \leq 104.$$

Note that the previously known lower bounds were reached by S. V. Chmutov (93 complex nodes: [2]) and D. van Straten (84 real nodes: a variant of Chmutov's construction using regular polygons instead of folding polynomials).

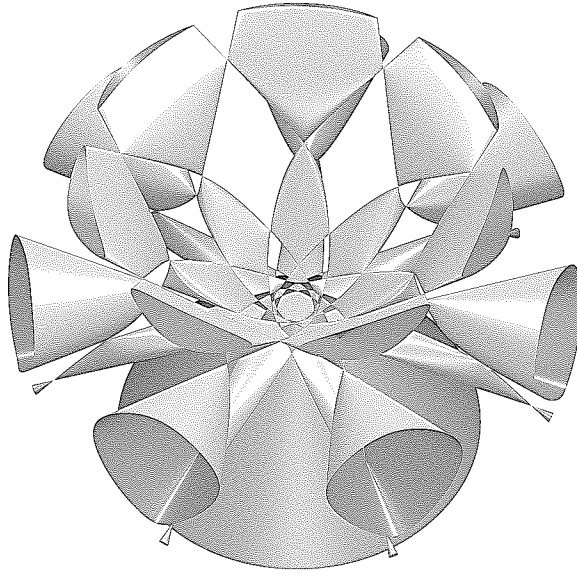


FIGURE 4. A part of the affine chart $w = 1$ of the real septic with 99 nodes, see [9] for more images and movies.

As it can be computed using deformation theory and SINGULAR that the space of obstructions for globalizing all local deformations is zero — this is based on ideas of D. van Straten, details will be published elsewhere — we obtain:

Corollary 3. *There exist surfaces of degree 7 in $\mathbb{P}^3(\mathbb{R})$ with exactly k real nodes and no other singularities for $k = 0, 1, 2, \dots, 99$.*

Recently there has been some interest in surfaces that do exist over some finite fields, but which are not liftable to characteristic 0. The reduction of our 99-nodal septic S_α modulo 5 (note: $1 \in \mathbb{F}_5$ satisfies (10): $7 \cdot 1^3 + 7 \cdot 1 + 1 \equiv 0$ modulo 5) neither gives a 99-nodal surface nor a highly degenerated one as one might expect because the exponent 5 appears several times in the defining equation. Instead, we can easily verify the following using computer algebra:

Corollary 4. *For $\alpha_5 := 1 \in \mathbb{F}_5$ the surface $S_{\alpha_5} \subset \mathbb{P}^3(\mathbb{F}_5)$ defined as in the above theorem has 100 nodes and no other singularities.*

Of course, not all the coordinates of its singularities are in \mathbb{F}_5 , but in some algebraic extension. The septic has similar geometric properties as our 99-nodal surface; in addition it has one node at the intersection of the $\{x = y = 0\}$ axes and $\{w = 0\}$. Until now, we were not able to determine if this 100-nodal septic defined over \mathbb{F}_5 can be lifted to characteristic zero.

We hope to be able to apply our technique for finding surfaces with many nodes within families of surfaces to similar problems. E.g., it should be possible to study surfaces with dihedral symmetry of degree 9 and 11 with many ordinary double points using the same ideas. Another application could be the search for surfaces with many cusps. We already studied families of such surfaces successfully using computer algebra in simpler cases [10].

APPENDIX: THE SINGULAR CODE TO PROVE THE THEOREM

```

proc isTrue(int c) { if(c==0) { return("FALSE"); } else { return("TRUE"); } }

ring r = (0,alpha), (x,y,z,w), dp; minpoly = 7*alpha^3 + 7*alpha + 1;
number a(1) = -12/7*alpha^2 - 384/49*alpha - 8/7;
number a(2) = -32/7*alpha^2 + 24/49*alpha - 4;
number a(3) = -4*alpha^2 + 24/49*alpha - 4;
number a(4) = -8/7*alpha^2 + 8/49*alpha - 8/7;
number a(5) = 49*alpha^2 - 7*alpha + 50;
poly P = x*(x^6-3*7*x^4*y^2+5*7*x^2*y^4-7*y^6)
        + 7*z*((x^2+y^2)^3-2^3*z^2*(x^2+y^2)^2+2^4*z^4*(x^2+y^2)) - 2^6*z^7;
poly U = (z+a(5)*w)*(a(1)*z^3+a(2)*z^2*w+a(3)*z*w^2+a(4)*w^3+(z+w)*(x^2+y^2))^2;
poly S = P-U;

"1. Check that the point (1:i:0:0) is not on S:";
poly Si = subst(subst(subst(S, x,1), z,0), w,0);
ideal yi = y^2+1; yi = std(yi);
isTrue(reduce(Si, yi) != 0), ": S(1,i,0,0) =", reduce(Si, yi); "";
"2. Check that there is exactly one node on the x=y=0 axes:";
ideal jSxy = x, y, jacob(S); jSxy = std(jSxy);
isTrue(mult(jSxy)==1), ": milnor =", mult(jSxy), ", dim =", (dim(jSxy)-1); "";
"3. Check that milnor(S_y) = 15 (takes some minutes):";
ideal jSy = y, jacob(S); jSy = std(jSy);
isTrue(mult(jSy)==15), ": milnor =", mult(jSy), ", dim =", (dim(jSy)-1); "";
"4. Switch to the affine chart w=1."; "";
S = subst(S,w,1);
ring raff = (0,alpha), (x,y,z), dp; minpoly = 7*alpha^3 + 7*alpha + 1;
poly S = imap(r,S);
"5. Check that all the sing. are in the affine chart w=1 (takes some minutes):";
ideal jSyaff = y, S, jacob(S); jSyaff = std(jSyaff);
isTrue(mult(jSyaff)==15), ": milnor =", mult(jSyaff), ", dim =", dim(jSyaff); "";
"6. Check that all the singularities are nodes (takes approx. an hour):";
poly hessian = det(jacob(jacob(S)));
ideal nonnodes = y, S, jacob(S), hessian; nonnodes = std(nonnodes);
isTrue(mult(nonnodes)==0), ": milnor =", mult(nonnodes), ", dim =", dim(nonnodes); "";}

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